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V Grave readings

Professor Dmytro Grave (September 6, 1863 – December 19, 1939) was the first academician of the Ukrainian Academy of Sciences in the field of mathematics, the founder of scientific and educational mathematical institutions, one of the founders of the Academy of Sciences, and one of the most prominent scientists whose work is associated with Ukraine.

V Grave readings

Locally matrix algebras and their applications

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Locally matrix algebras: definition

Let $\mathbb{F}$ be a ground field. All vector spaces are considered over $\mathbb{F}$. Denote by $M_n(\mathbb{F})$ the algebra of $n \times n$ matrices over $\mathbb{F}$.

Finite–dimensional algebras $\Longrightarrow$ infinite–dimensional algebras.

"Locally semisimple algebras are... next in complexity after finite–dimensional algebras." *A. Vershik, S. Kerov*

Definition

An associative algebra A is called a **locally matrix algebra** if for each finite subset of A there exists a subalgebra $B \subseteq A$ containing this subset, such that $B \cong M_n(\mathbb{F})$ for some $n \geq 1$. The algebra A is **unital** if $A \ni 1$.

Content of the talk

The talk consists of the following parts:

I. **Locally matrix algebras: decompositions**
II. **Automorphisms and derivations of locally matrix algebras**
III. **Clifford algebras**
IV. **Algebras of infinite matrices**
V. **Mackey algebras and groups**
VI. **Derivations of polynomial algebras in infinitely many variables**

I. Locally matrix algebras: decompositions

[G. Köthe, 1931]

Every countable-dimensional unital locally matrix algebra A is isomorphic to a tensor product of matrix algebras:

$$A = \bigotimes_{i=1}^{\infty} A_i, \quad A_i \cong M_{n_i}(\mathbb{F}).$$

[A. Kurosh, 1942]

*An example of an uncountable-dimensional unital locally matrix algebra that is **not** isomorphic to any tensor product of matrix algebras.*

I. Locally matrix algebras: decompositions

Definition

A locally matrix algebra A is called **primary** if there exists a prime p such that every finite subset of A lies in a subalgebra isomorphic to $M_{p^n}(\mathbb{F})$ for some $n \geq 1$.

[V. Kurochkin, 1948]

Studied decompositions into tensor products of primary algebras.

Question [V. Kurochkin, 1948]

Is every unital locally matrix algebra decompose into a tensor product of primary algebras?

For unital countable-dimensional locally matrix algebras, it follows from Köthe's theorem:

Corollary

Every unital countable-dimensional locally matrix algebra admits a decomposition into a tensor product of primary algebras.

Remark

1. **Kurosh's example of an uncountable-dimensional unital locally matrix algebra is a primary algebra for prime 2.**
2. **Clifford algebra is a primary algebra, and the generalized Clifford algebra for prime positive integer is also a primary algebra.**

I. Locally matrix algebras: decompositions

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of positive numbers, and let $\mathbb{P} = \{2, 3, 5, \dots\}$ be the set of all primes.

Definition (E. Steinitz, 1910)

A **Steinitz (supernatural) number** is a formal product $\mathrm{s} = \prod_{p \in \mathbb{P}} p^{r_p}$, where each exponent $r_p \in \mathbb{N} \cup \{0, \infty\}$. Steinitz numbers are multiplied by the rule: $\prod_{p \in \mathbb{P}} p^{r_p} \cdot \prod_{p \in \mathbb{P}} p^{k_p} = \prod_{p \in \mathbb{P}} p^{r_p + k_p}$.

Definition (O. B. – B. Oliynyk, 2020)

Let A be a unital locally matrix algebra over a field $\mathbb{F}$. Let

$$D(A) = \{n \in \mathbb{N} \,|\, \text{there exists} \quad 1 \in B \subset A, \quad B \cong M_n(\mathbb{F})\}.$$

The least common multiple $\mathrm{st}(A) = \operatorname{lcm} D(A)$ is called the **Steinitz number of the algebra** A.

Remark

Let A be a unital countable-dimensional locally matrix algebra, $A = \bigotimes_{i=1}^{\infty} M_{n_i}(\mathbb{F})$. Then $\mathrm{st}(A) = \prod_{i=1}^{\infty} n_i$.

[J. Glimm, 1960]

Let A and B be unital countable-dimensional locally matrix algebras over a field $\mathbb{F}$. Then

$$A \cong B \iff \mathrm{st}(A) = \mathrm{st}(B).$$

[O. B. – B. Oliynyk – V. Sushchansky, 2016]

This approach was extended to many other classes of countable-dimensional structures.

I. Locally matrix algebras: decompositions

Theorem 1 [O. B. – B. Oliynyk, 2020]

1. *In the uncountable-dimensional case, the Steinitz number* $\mathrm{st}(A)$ *does not determine the unital locally matrix algebra* A*, but it determines the universal elementary theory of* A*.*
2. *There exists a unital locally matrix algebra that is not decomposable into a tensor product of primary algebras.*

Question [V. Kurochkin, 1948]

Theorem 1(2) gives a negative answer to V. Kurochkin's question.

This part is based on papers:

[1] O.B., B.Oliynyk, Primary decompositions of unital locally matrix algebras, *Bulletin of Mathematical Sciences*, **10**(1) (2020). **Q1**

[2] O.B., B.Oliynyk, Unital locally matrix algebras and Steinitz numbers, *Journal of Algebra and Its Applications*, **19**(9) (2020). **Q2**

Let A be an algebra over a field $\mathbb{F}$.

Definition

A linear bijective mapping $\varphi\colon A \to A$ is called an **automorphism** if

$$\varphi(ab) = \varphi(a)\,\varphi(b) \qquad \text{for all elements } a, b \in A.$$

The set of all automorphisms of A forms a group, denoted $\mathrm{Aut}(A)$.

This was studied by **É. Galois**, **F. Klein**, and many other authors.

Definition

A linear mapping $d\colon A \to A$ is called a **derivation** if

$$d(ab) = d(a)\,b + a\,d(b) \qquad \text{for all elements } a, b \in A.$$

The set of all derivations of A forms a Lie algebra, denoted $\mathrm{Der}(A)$.

Example (1)

Let L be a Lie algebra with operation $[x, y]$, and let $a \in L$. The linear mapping

$$\mathrm{ad}(a) : L \to L, \quad \mathrm{ad}(a) : x \to [a, x] \quad (x \in L),$$

is a derivation of L.

Example (2)

Let A be an associative algebra, and let $a \in A$. The linear mapping

$$\mathrm{ad}(a) : A \to A, \quad \mathrm{ad}(a) : x \to [a, x] = ax - xa \quad (x \in A),$$

is a derivation of A. Indeed, $A^{(-)} = \big(A,\ [a, b] = ab - ba\big)$ is a Lie algebra.

Definition

Derivations from Examples (1) and (2) are called **inner derivations**.

Example (3)

Let A be an algebra. The conjugation $\varphi\colon A \to A$, $\varphi_g : x \to gxg^{-1}$, by the invertible elements $g \in A$ is an automorphism of A.

Definition

Automorphisms from Examples (3) are called **inner automorphisms**.

Remark

Let $d : A \to A$ be a derivation. Suppose the the exponential series

$$\varphi = \exp(d) = \mathrm{Id} + d + \frac{d^2}{2!} + \cdots$$

makes sense (topological or any other). Then φ is an automorphism. Hence,

derivations are "infinitesimal automorphisms"

Let's denote:

Der(A) – the Lie algebra of all derivations of A.

Inder(A) – the Lie algebra of inner derivations is an ideal in Der(A).

Outder(A) = Der(A)/Inder(A) – the Lie algebra of outer derivations.

We will describe all automorphisms and derivations of the unital countable-dimensional locally matrix algebra

$$A = \bigotimes_{i=1}^{\infty} A_i, \qquad A_i \cong M_{n_i}(\mathbb{F}).$$

[H. Strade, 1999]

Studied derivations of countable-dimensional (diagonal) locally simple Lie algebras over a field of characteristic 0.

Question

How large is the Lie algebra of outer derivations $\mathrm{Outder}(A)$*?*

[S. Ayupov – K. Kudaybergenov, 2020]

The algebra $A = \otimes_{i=1}^{\infty} M_{n_i}(\mathbb{F})$ *has an outer derivation.*

We describe derivations of the algebra $A = \otimes_{i=1}^{\infty} A_i,\ A_i \cong M_{n_i}(\mathbb{F})$.

Definition

Let $\mathcal{P}$ be a system of nonempty finite subsets of $\mathbb{N}$. We say that $\mathcal{P}$ is **sparse** if:

1. for any $S \in \mathcal{P}$, all nonempty subsets of S also lie in $\mathcal{P}$,
2. an arbitrary element $i \in \mathbb{N}$ lies in no more then finitely many subsets $S \in \mathcal{P}$.

If $S = \{i_1, \ldots, i_r\}$, then denote

$$A_S = A_{i_1} \otimes \cdots \otimes A_{i_r}.$$

Choose for each $S \in \mathcal{P}$ an element $a_S \in A_S$. Then the (possibly infinite) sum

$$\sum_{S \in \mathcal{P}} \mathrm{ad}(a_S)$$

converges (in the Tychonoff topology) to a derivation of A.

Fix a sparse system $\mathcal{P}$. Consider the vector space of all infinite convergent sums of inner derivations:

$$D_{\mathcal{P}} = \Big\{ \sum_{S \in \mathcal{P}} \mathrm{ad}(a_S) \;\Big|\; a_S \in A_S,\; S \in \mathcal{P} \Big\}.$$

In each subalgebra A_i, choose a subspace $A_i^0 \subset A_i$ so that

$$A_i = \mathbb{F} \cdot 1_{A_i} \oplus A_i^0 \quad \text{(direct sum)}.$$

In each A_i^0 fix a basis $\mathcal{E}_i$, and for each finite $S = \{i_1, \ldots, i_r\} \in \mathcal{P}$, let

$$\mathcal{E}_S = \mathcal{E}_{i_1} \otimes \cdots \otimes \mathcal{E}_{i_r} = \{ e_{i_1} \otimes \cdots \otimes e_{i_r} \mid e_{i_j} \in \mathcal{E}_{i_j} \}.$$

Theorem 2 [O. B., 2021]

1. *The Lie algebra of all derivations of* $A = \otimes_{i=1}^{\infty} M_{n_i}(\mathbb{F})$ *is given by*

$$\mathrm{Der}(A) = \bigcup_{\mathcal{P}} D_{\mathcal{P}},$$

   where the union runs over all sparse systems $\mathcal{P}$*.*

2. *The set*

$$\bigcup_{S \in \mathcal{P}} \mathrm{ad}(\mathcal{E}_S)$$

   is a topological basis of the vector space $D_{\mathcal{P}}$*.*

3. *The Lie algebra* $\mathrm{Inder}(A)$ *is dense in* $\mathrm{Der}(A)$ *in the Tykhonov topology.*

Theorem 3 [O. B., 2021]

Let A be a countable-dimensional locally matrix algebra. Then

$$\dim_{\mathbb{F}} \operatorname{Der}(A) = \dim_{\mathbb{F}} \operatorname{Outder}(A) = |\mathbb{F}|^{\aleph_0}.$$

Remark

So, the Lie algebra of (outer) derivations is not too small.

This part is based on the paper:

[1] O.B., Derivations and automorphisms of locally matrix algebras, *Journal of Algebra*, **576** (2021), 1-26. **Q1**

We describe automorphisms of the algebra $A = \otimes_{i=1}^{\infty} A_i$, $A_i \cong M_{n_i}(\mathbb{F})$.

Let $H_n = \{\text{conjugations by invertible elements from } A_n \otimes A_{n+1} \otimes \cdots\}$.
Then

$$H_1 > H_2 > \cdots, \qquad \bigcap_{n \geq 1} H_n = (1).$$

Choose for each $n \geq 1$ a system X_n of representatives of left cosets of the subgroup H_{n+1} in H_n, so

$$H_n = \bigsqcup_{x \in X_n} x H_{n+1}, \quad \text{and we assume} \quad 1 \in X_n.$$

Every infinite product $\varphi = \varphi_1 \varphi_2 \varphi_3 \cdots$, where $\varphi_n \in X_n$, converges (in the Tykhonov topology) to an injective endomorphism $\varphi \colon A \to A$.
We write succinctly $\varphi = \varphi_1 \varphi_2 \cdots$.

Theorem 4 [O. B., 2021]

Every unital injective endomorphism $\varphi\colon A \to A$ *can be uniquely represented as*

$$\varphi = \varphi_1\,\varphi_2 \cdots, \quad \textit{where} \quad \varphi_n \in X_n,\ n \geq 1.$$

Moreover, **there are simple necessary and sufficient conditions** on the sequence φ_n, $n \geq 1$, under which φ **is in fact an automorphism**.

Theorem 5 [O. B., 2021]

Let A be a countable-dimensional locally matrix algebra. Then

$$\left|\mathrm{Aut}(A)\right| = \left|\mathbb{F}\right|^{\aleph_0}.$$

This part is based on the paper:

[1] O.B., Derivations and automorphisms of locally matrix algebras, *Journal of Algebra*, **576** (2021), 1-26. **Q1**

Clifford algebra was first defined by William Clifford in the late 19th century.

In 1878, Clifford greatly expanded on Grassmann's work to the form what are now usually called Clifford algebras in his honor. Clifford himself chose to call them as ***geometric algebras***.

Definition

Let $\mathbb{F}$ be a field of characteristic not equal to 2. Let V be a vector space over $\mathbb{F}$. A mapping $f : V \times V \to \mathbb{F}$ is called a **quadratic form** if
(1) $f(\lambda v) = \lambda^2 f(v)$,
(2) $f(v, w) = f(v + w) - f(v) - f(w)$ is a bilinear form.
A quadratic form f is **nondegenerate** if the bilinear form $f(v, w)$ is nondegenerate.

Definition

The **Clifford algebra** $\mathcal{C}\ell(V, f)$ is generated by the vector space V and unit 1 with defining relations

$$v^2 = f(v) \cdot 1 \quad \text{for} \quad v \in V.$$

If $\{v_i\}_{i \in I}$ is a basis of the vector space V and the set of indices I is ordered, then the set of ordered products

$$v_{i_1} \cdots v_{i_k}, \quad \text{where} \quad i_1 < i_2 < \ldots < i_k, \quad \text{and } 1,$$

form a basis of the Clifford algebra $\mathcal{C}\ell(V, f)$.

III. Clifford algebras

The Clifford algebra $\mathcal{C}\ell(V,f)$ is graded by the cyclic group of order 2, expressed as the sum of even and odd components:

$$\mathcal{C}\ell(V,f) = \mathcal{C}\ell(V,f)_{\bar{0}} + \mathcal{C}\ell(V,f)_{\bar{1}}, \quad \text{where}$$

$$\mathcal{C}\ell(V,f)_{\bar{0}} = \mathbb{F}\cdot 1 + \sum_{n=1}^{\infty} \underbrace{V\cdots V}_{2n}, \quad \mathcal{C}\ell(V,f)_{\bar{1}} = \sum_{n=0}^{\infty} \underbrace{V\cdots V}_{2n+1}.$$

[N. Jacobson, 1968]

Suppose that the field $\mathbb{F}$ is algebraically closed, and the quadratic form f is nondegenerate. If $\dim_{\mathbb{F}} V = d$ is an even integer, then the Clifford algebra

$$\mathcal{C}\ell(V,f) \cong M_{2^{\frac{d}{2}}}(\mathbb{F}) \quad \text{of} \quad 2^{\frac{d}{2}} \times 2^{\frac{d}{2}} \quad \text{matrices over } \mathbb{F}.$$

If d is odd, then $\mathcal{C}\ell(V,f) \cong M_{2^{\frac{d-1}{2}}}(\mathbb{F}) \oplus M_{2^{\frac{d-1}{2}}}(\mathbb{F}).$

III. Clifford algebras

Very little is known about Clifford algebras $\mathcal{C}\ell(V, f)$ of infinite-dimensional vector spaces V.

The motivation comes from ***mathematical physics*** and $\mathbb{C}^*$-***algebras***, for example, in the sense of embedding $\mathcal{C}\ell(V, f) \hookrightarrow \mathbb{C}^*$-algebra.

Remark

In particular, if $\mathbb{F} = \mathbb{R}$ and the quadratic form f is positive-definite, then $\mathcal{C}\ell(V, f)$ is a normed algebra.

Theorem 6 [O. B. – B. Oliynyk, 2021]

If V is infinite-dimensional over an algebraically closed field and f is nondegenerate, then $\mathcal{C}\ell(V, f)$ is a unital locally matrix algebra with

$$\mathrm{st}\left(\mathcal{C}\ell(V, f)\right) \;=\; 2^{\infty}.$$

Let $\dim_{\mathbb{F}} V = \aleph_0$. By Köthe's theorem, one then obtains a tensor-product decomposition

$$\mathcal{C}\ell(V, f) \cong \bigotimes_{i=1}^{\infty} \Big(\text{finite-dimensional matrix algebra}\Big).$$

We will now present this decomposition explicitly.

Decomposition of an infinite-dimensional Clifford algebra.

Suppose that the field $\mathbb{F}$ is algebraically closed and the vector space V is countably-dimensional. Choose an orthonormal basis

$$\{v_i\}_{i\in\mathbb{N}} \subset V \quad \text{so that} \quad v_i v_j + v_j v_i = 2\,\delta_{ij}, \quad i,j \in \mathbb{N}.$$

Let

$$0 = n_0 < n_1 < n_2 < \cdots$$

be any strictly increasing sequence of *even* integers, and set for each $i \geq 1$

$$V_i \;=\; \mathrm{Span}\big(v_{n_{i-1}+1},\, \ldots,\, v_{n_i}\big).$$

III. Clifford algebras

The subalgebra of $C\ell(V,f)$ generated by V_i is isomorphic to $C\ell(V_i,f)$, and it is $\mathbb{Z}/2\mathbb{Z}$–graded:

$$C\ell(V_i,f) = C\ell(V_i,f)_{\overline{0}} + C\ell(V_i,f)_{\overline{1}}.$$

Let $c_i = v_1 \cdots v_{n_i}$. Consider the subalgebras

$$A_1 = C\ell(V_1,f), \quad A_i = C\ell(V_i,f)_{\overline{0}} \oplus c_i\, C\ell(V_i,f)_{\overline{1}} \quad (i \geq 2).$$

Theorem 7 [O. B., 2025]

$A_i \cong C\ell(V_i,f)$ *for each* $i \in \mathbb{N}$; $[A_i, A_j] = (0)$ *for* $i, j \in N$, $i \neq j$; *and*

$$C\ell(V,f) \cong \bigotimes_{i\in\mathbb{N}} A_i.$$

Up to now, the only examples of derivations and automorphisms of $\mathcal{C}\ell(V, f)$ in the literature are:

1. inner derivations and inner automorphisms,
2. Bogolyubov derivations and Bogolyubov automorphisms.

III. Clifford algebras

Definition

Let φ be an invertible linear transformation $\varphi : V \to V$ that preserves the quadratic form,

$$f(\varphi(v)) = f(v)$$

for an arbitrary element $v \in V$. It is easy to see that φ uniquely extends to an automorphism of the algebra $\mathcal{C}\ell(V, f)$. Such automorphisms are called **Bogolyubov automorphisms**.

Definition

Let $\psi : V \to V$ be a skew-symmetric linear transformation, that is,

$$f(\psi(v), w) + f(v, \psi(w)) = 0 \quad \text{for all} \quad v, w \in V.$$

The mapping ψ uniquely extends to a derivation of the algebra $\mathcal{C}\ell(V, f)$. These derivations are called **Bogolyubov derivations**.

Definition

A **derivation** D of the algebra $\mathcal{C}\ell(V, f)$ is called **even** if

$$D(\mathcal{C}\ell(V, f)_{\bar{0}}) \subseteq \mathcal{C}\ell(V, f)_{\bar{0}}, \quad D(\mathcal{C}\ell(V, f)_{\bar{1}}) \subseteq \mathcal{C}\ell(V, f)_{\bar{1}};$$

and a **derivation** D of the algebra $\mathcal{C}\ell(V, f)$ is called **odd** if

$$D(\mathcal{C}\ell(V, f)_{\bar{0}}) \subseteq \mathcal{C}\ell(V, f)_{\bar{1}}, \quad D(\mathcal{C}\ell(V, f)_{\bar{1}}) \subseteq \mathcal{C}\ell(V, f)_{\bar{0}}.$$

Remark

Note that Bogolyubov derivations of Clifford algebras are even.

Let $S = \{i_1 < \cdots < i_r\}$ be a set of positive integers. Denote $v_S = v_{i_1} \cdots v_{i_r}$.

The following theorem describes all derivations of $\mathcal{C}\ell(V, f)$ and creates many new derivations (neither inner nor Bogolyubov).

Theorem 8 [O. B., 2025]

Let V be a countable-dimensional vector space over an algebraically closed field $\mathbb{F}$, and let v_i, $i \in \mathbb{N}$, be an arbitrary orthonormal basis of the space V.

(1) *Any nonzero even derivation D of $\mathcal{C}\ell(V, f)$ can be uniquely represented as a sum* $D = \sum_S \alpha_S \operatorname{ad}(v_S)$, $0 \neq \alpha_S \in \mathbb{F}$, *where the subsets S are finite nonempty subsets of $\mathbb{N}$ of even order, and any $i \in \mathbb{N}$ lies in no more than finitely many subsets S.*

(2) *Any nonzero odd derivation D of $\mathcal{C}\ell(V, f)$ can be uniquely represented as a sum* $D = \sum_S \alpha_S \operatorname{ad}(v_S)$, $0 \neq \alpha_S \in \mathbb{F}$, *where the subsets S are finite subsets of $\mathbb{N}$ of odd order, and any $i \in \mathbb{N}$ lies in all but finitely many subsets S.*

Remark

1. Notice that a **derivation D is inner if and only if** the sum (1) or (2) above is finite.
2. A **derivation D is Bogolyubov if and only if**

$$D = \sum_{i<j} \alpha_{ij} \operatorname{ad}(v_i v_j), \quad \alpha_{ij} \in \mathbb{F},$$

where for each $i \in \mathbb{N}$ only finitely many coefficients α_{ij}, $i < j$, are nonzero.

Suppose the ground field is the field of real numbers $\mathbb{R}$, and let $f : V \to \mathbb{R}$ be a positive-definite quadratic form.

Then $\mathcal{C}\ell(V, f)$ is a normed algebra.

The completion of $\mathcal{C}\ell(V, f)$ is a $\mathbb{C}^*$-algebra.

Question [M. Ludewig, MathOverflow discussion 2022]

Is an arbitrary automorphism of the Clifford algebra continuous?

Using the decomposition of Theorem 7:

Theorem 9 [O. B., 2025]

*There exists an automorphism of $\mathcal{C}\ell(V, f)$ that is **not** continuous with respect to the topology induced by the norm.*

Remark

Theorem 9 gives negative answer to the question of M. Ludewig.

This part is based on the paper:

[1] O.B., On Clifford Algebras of Infinite Dimensional Vector Spaces, Proceedings XIV Ukraine Algebra Conference, *Contemporary Math. AMS* (2025). **Q3**

IV. Algebras of infinite matrices

Let $M_n(\mathbb{F})$ denote the algebra of $n \times n$ matrices over the field $\mathbb{F}$. Consider the ascending chain of matrix algebras over a field $\mathbb{F}$ with embeddings:

$$M_2(\mathbb{F}) \hookrightarrow M_3(\mathbb{F}) \hookrightarrow \cdots M_n(\mathbb{F}) \hookrightarrow \cdots, \quad M_n(\mathbb{F}) \to \begin{pmatrix} * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in M_{n+1}(\mathbb{F}).$$

Denote the union of of such matrix algebras

$$M_\infty(\mathbb{F}) = \bigcup_{n \geq 1} M_n(\mathbb{F}) =$$

$$\{ \text{infinite } \mathbb{N} \times \mathbb{N} \text{ matrices having finitely many nonzero entries} \}.$$

IV. Algebras of infinite matrices

Consider the algebra $M_{\mathbb{N}}(\mathbb{F}) = \{\mathbb{N} \times \mathbb{N}$ matrices having

finitely many nonzero entries in each column$\}$.

Then $M_\infty(\mathbb{F}) \subset M_{\mathbb{N}}(\mathbb{F})$.

Let $M_{\text{fin}}(\mathbb{F}) = \{\mathbb{N} \times \mathbb{N}$ matrices having finitely many nonzero entries

in each row and each column$\}$.

Then $M_\infty(\mathbb{F}) \lhd M_{\text{fin}}(\mathbb{F}) \subset M_{\mathbb{N}}(\mathbb{F})$.

Remark

$M_\infty(\mathbb{F})$ **is a locally matrix algebra**. In turn, the description of derivations and automorphisms of locally matrix algebras is **extensively used** in the study of derivations and automorphisms of algebras of infinite matrices.

Each associative algebra A gives rise to the Lie algebra

$$A^{-} = (A, [a, b] = ab - ba).$$

Hence, denoting by $\mathfrak{gl}_{\infty}(\mathbb{F})$, $\mathfrak{gl}_{\text{fin}}(\mathbb{F})$ and $\mathfrak{gl}_{\mathbb{N}}(\mathbb{F})$ the corresponding Lie algebras of the algebras $M_{\infty}(\mathbb{F})$, $M_{\text{fin}}(\mathbb{F})$, and $M_{\mathbb{N}}(\mathbb{F})$, we obtain

$$\mathfrak{gl}_{\infty}(\mathbb{F}) \lhd \mathfrak{gl}_{\text{fin}}(\mathbb{F}) \subset \mathfrak{gl}_{\mathbb{N}}(\mathbb{F}).$$

Definition

Recall that

$$\mathfrak{sl}_{\infty}(\mathbb{F}) = [\mathfrak{gl}_{\infty}(\mathbb{F}), \mathfrak{gl}_{\infty}(\mathbb{F})].$$

IV. Algebras of infinite matrices

The algebra $M_\infty(\mathbb{F})$ is equipped with the **transpose involution**

$$t : (a_{ij}) \mapsto (a_{ji}),$$

and, on $M_2(\mathbb{F})$ in particular, also the **symplectic involution**

$$^- : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

Therefore, we have $M_\infty(\mathbb{F}) \cong M_\infty(M_2(\mathbb{F}))$, $\mathrm{sp} : (a_{ij}) \to (\overline{a_{ji}})$.

This gives rise to Lie algebras of skew-symmetric elements

$$\mathfrak{o}_\infty(\mathbb{F}) = \{a \in M_\infty(\mathbb{F}) \mid a^t = -a\},$$

$$\mathfrak{sp}_\infty(\mathbb{F}) = \{a \in M_\infty(\mathbb{F}) \mid \mathrm{sp}(a) = -a\}.$$

We obtain the infinite **orthogonal and symplectic Lie algebras**.

IV. Algebras of infinite matrices

We will describe derivations and automorphisms of the following Lie algebras:

$$\mathfrak{sl}_\infty(\mathbb{F}), \quad \mathfrak{o}_\infty(\mathbb{F}), \quad \mathfrak{sp}_\infty(\mathbb{F}), \quad \mathfrak{gl}_{\text{fin}}(\mathbb{F}), \quad \mathfrak{gl}_{\mathbb{N}}(\mathbb{F}).$$

In the following two theorems we assume that $\operatorname{char}\mathbb{F} \neq 2$.

Theorem 10 [O. B., 2022]

(a) *An arbitrary derivation of the Lie algebra $\mathfrak{sl}_\infty(\mathbb{F})$, $\mathfrak{o}_\infty(\mathbb{F})$ or $\mathfrak{sp}_\infty(\mathbb{F})$ is of the type* $\mathrm{ad}(a)$, *where* $a \in \mathfrak{gl}_{fin}(\mathbb{F})$. *For $\mathfrak{o}_\infty(\mathbb{F})$ or $\mathfrak{sp}_\infty(\mathbb{F})$ the element a is skew-symmetric relative to the corresponding involution.*

(b) *All derivations of the Lie algebras $\mathfrak{gl}_{fin}(\mathbb{F})$ and $\mathfrak{gl}_{\mathbb{N}}(\mathbb{F})$ are inner.*

Remark

Theorem 10(a) for fields of characteristic 0 is due to *K.-H. Neeb (2005)*.

IV. Algebras of infinite matrices

Consider the group $GL_{\text{fin}}(\mathbb{F}) = \{ a \in GL_{\mathbb{N}}(\mathbb{F}) \mid a, a^{-1} \in M_{\text{fin}}(\mathbb{F})\}$.

Theorem 11 [O. B., 2022]

(a) *An arbitrary automorphism φ of the Lie algebra $L = \mathfrak{sl}_\infty(\mathbb{F})$ is of the type $\varphi(x) = a^{-1}xa$, $x \in L$, or of the type $\varphi(x) = -a^{-1}x^t a$, $x \in L$, where $a \in GL_{\text{fin}}(\mathbb{F})$.*

(b) *An arbitrary automorphism φ of the orthogonal (resp. symplectic) Lie algebra $L = \mathfrak{o}_\infty(\mathbb{F})$ (resp. $\mathfrak{sp}_\infty(\mathbb{F})$) is of the type $\varphi(x) = a^{-1}xa$, $x \in L$, where $a \in GL_{\text{fin}}(\mathbb{F})$ and a is orthogonal (resp. symplectic) with respect to the defining involution.*

(c) *All automorphisms of $\mathfrak{gl}_{\text{fin}}(\mathbb{F})$ and of $\mathfrak{gl}_{\mathbb{N}}(\mathbb{F})$ are given by conjugation by elements of $GL_{\text{fin}}(\mathbb{F})$ and $GL_{\mathbb{N}}(\mathbb{F})$, respectively.*

Remark

Theorem 11(a),(b) for fields of characteristic 0 is due to *N. Stumme (2001)*.

***K.-H. Neeb and N. Stumme* relied on the representation theory of finite-dimensional simple Lie algebras over a field of characteristic 0.**

In positive characteristic, that theory becomes much more complicated.
Theorems 10 and 11 for fields of characteristic $\neq 2$ was proved in:
*O. B., Automorphism and derivations of algebras of infinite matrix, Linear Algebra and Applications, **650**(2) (2022), p.42-59.* **Q1**
Our approach consists of two steps:

1. **Representation of automorphisms of Lie algebras as combinations of homomorphisms and anti-homomorphisms of the underlying associative algebras.** Here we rely on a series of papers by *K. Beidar, M. Brešar, M. Chebotar and W. Martindale*, which prove Herstein's conjectures.

2. **Description of automorphisms and anti-automorphisms of the underlying associative algebras.**

IV. Algebras of infinite matrices

In the second step, characteristic is inelevant, but *K. Beidar, M. Brešar, M. Chebotar and W. Martindale* assumed that the characteristic is $\neq 2$. **This was the main difficulty, which was overcome in the joint paper:**

O. B, I. Kashuba, E. Zelmanov, On Lie isomorphisms of rings, *Mediterranean Journal of Mathematics* (2025) 22:80, Q2

which handled Lie isomorphisms of rings of arbitrary characteristic, not necessarily unital.

We will talk about Mackey algebras.

Let V be an infinite-dimensional vector space, and let $V^* = \{ f : V \to F \mid f \text{ is linear functionals}\}$ be its dual space.

Definition

A subspace $W \subseteq V^*$ is called **total** if for all $v \in V$

$$(v \mid W) = \{ w(v) \mid w \in W\} = (0) \iff v = 0.$$

For a total subspace $W \subseteq V^*$, consider the subalgebra

$$A(V \mid W) = \{\varphi \in \mathrm{End}_{\mathbb{F}}(V) \mid W\varphi \subseteq W\}.$$

V. Mackey algebras and groups

Definition

A linear transformation $\varphi \in \mathrm{End}_F(V)$ is called **finitary** if $\dim_F \varphi(V) < \infty$.

Consider $$A_\infty(V\,|\,W) \;=\; \{\varphi \in A(V\,|\,W) \mid \varphi \text{ is finitary}\}.$$

Remark

The algebra $A_\infty(V\,|\,W)$ is a nonunital locally matrix. In turn, the description of derivations and automorphisms of locally matrix algebras is *extensively used* in the study of derivations and automorphisms of Mackey algebras and groups.

V. Mackey algebras and groups

[G. Mackey, 1945]

If $\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} W = \aleph_0$, then:

1. $A_{\infty}(V \mid W)$ is isomorphic to the associative algebra $M_{\infty}(\mathbb{F})$ of countable matrices over a field $\mathbb{F}$ having finitely many nonzero entries;
2. the algebra $A(V \mid W)$ is isomorphic to the associative algebra $M_{fin}(\mathbb{F})$ of countable matrices over a field $\mathbb{F}$ having finitely many nonzero entries in each row and in each column.

Remark

If total subspace $W = V^*$, then

$$A(V \mid W) = \mathrm{End}_{\mathbb{F}}(V),$$

$$A_{\infty}(V \mid W) = \{\text{all transformations of finite range}\}.$$

Clearly,

$$A_\infty(V \mid W) \triangleleft A(V \mid W).$$

The algebra $A_\infty(V \mid W)$ gives rise to Lie algebras

$$\mathfrak{gl}_\infty(V \mid W) \;=\; \Big(A_\infty(V \mid W),\ [\varphi, \psi] = \varphi\psi - \psi\varphi\Big) \quad \text{and}$$

$$\mathfrak{sl}_\infty(V \mid W) \;=\; [\mathfrak{gl}_\infty(V \mid W), \mathfrak{gl}_\infty(V \mid W)].$$

Moreover, $\mathfrak{sl}_\infty(V \mid W)$ is an inductive limit of the finite-dimensional simple Lie algebras $\mathfrak{sl}(n)$:

$$\mathfrak{sl}_\infty(V \mid W) \;=\; \varinjlim_n \mathfrak{sl}(n).$$

If V is equipped with a symmetric or skew-symmetric nondegenerate bilinear form, then we consider **orthogonal and symplectic Lie algebras** $\mathfrak{o}(V|W)$, $\mathfrak{sp}(V|W)$ and their **finitary versions** $\mathfrak{o}_\infty(V|W)$, $\mathfrak{sp}_\infty(V|W)$.

Definition

The algebras

$$A(V\,|\,W),\quad A_\infty(V\,|\,W),\quad \mathfrak{gl}_\infty(V\,|\,W),\quad \mathfrak{sl}_\infty(V\,|\,W),$$

$$\mathfrak{o}(V|W),\quad \mathfrak{sp}(V|W),\quad \mathfrak{o}_\infty(V|W),\quad \mathfrak{sp}_\infty(V|W)$$

are called **associative Mackey algebras** and **Lie Mackey algebras**, respectively.

V. Mackey algebras and groups

Mackey algebras and groups arise in various classifications.

Baranov–Strade classification [A. Baranov – H. Strade, 2002]

Proved that infinite-dimensional simple finitary Lie algebras over an algebraically closed field of characteristic $\neq 2, 3$ are:

1. $\mathfrak{sl}_\infty(V|W)$,
2. $\mathfrak{o}_\infty(V|W)$, $\mathfrak{sp}_\infty(V|W)$.

All algebras (1)-(2) are Mackey.

References:

[1] A.Baranov, H.Strade, Finitary Lie algebras, *Journal of Algebra*, **254**(1) (2002), 173–211.

V. Mackey algebras and groups

Let $\quad \mathrm{GL}(V) = \{\varphi \in \mathrm{End}_F(V) \mid \varphi \text{ is invertible}\}.$

Definition

Similarly, we define **Mackey groups** (of invertible elements):

$$\mathrm{GL}(V\,|\,W) = \{\varphi \in \mathrm{GL}(V) \mid \text{ both } \varphi \text{ and } \varphi^{-1} \text{ lie in } A(V\,|\,W)\},$$
$$\mathrm{GL}_\infty(V\,|\,W) = (\mathrm{Id} + A_\infty(V\,|\,W)) \cap \mathrm{GL}(V),$$
$$\mathrm{SL}_\infty(V\,|\,W) = [\mathrm{GL}_\infty(V\,|\,W), \mathrm{GL}_\infty(V\,|\,W)];$$

orthogonal and symplectic groups: $\boldsymbol{O}(V|W)$, $\boldsymbol{SP}(V|W)$,
their finitary versions: $\boldsymbol{O}_\infty(V|W)$, $\boldsymbol{SP}_\infty(V|W)$,
their special finitary versions: $\boldsymbol{SO}_\infty(V|W)$, $\boldsymbol{Sp}_\infty(V|W)$, and the corresponding special finitary unitary groups: $\boldsymbol{SU}_\infty(V|W)$, $\boldsymbol{SpU}_\infty(V|W)$.

Jonathan Hall's classification [J. Hall, 1995, 2006]

Proved that infinite simple finitary torsion groups are:

1. Alt[X], where X is an infinite set,
2. $\mathrm{SL}_\infty(V|W)$,
3. $SO_\infty(V|W)$, $Sp_\infty(V|W)$,
4. $SU_\infty(V|W)$, $SpU_\infty(V|W)$,

where ground fielf $\mathbb{F}$ is an algbraic extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

All groups (2)-(4) are Mackey.

References:

[1] J.Hall, Locally finite simple groups of finitary linear transformations, **in:** B.Hartley, G.M.Seitz, A.V.Borovik, R.M.Bryant (Eds.), Finite and Locally Finite Groups, Kluwer, Dordrecht, 1995, 219–246.

[2] J.Hall, Periodic simple groups of finitary linear transformations, *Annals of Mathematics*, **163** (2006), 445–498.

Description of the derivations of all infinite-dimensional simple finitary Lie algebras (Mackey algebras) due to Baranov-Strade classification (for fields of characteristic $\neq 2$).

Theorem 12 (on Mackey algebras) [O. B., 2023]

Let $\mathrm{char}\,\mathbb{F} \neq 2$. Derivations of each of the Mackey Lie algebras are adjoint derivations

$$\mathrm{ad}(a) : x \mapsto [a, x], \quad \textit{where } a \in A(V|W).$$

Description of the automorphisms of all infinite simple finitary torsion groups (Mackey groups) due to Hall's classification (for fields of characteristic $\neq 2$ or 3).

1. J. Schreier and S. Ulam (1933) described automorphisms of $\mathrm{Alt}[X]$, X an infinite set.

Theorem 13 (on Mackey groups) [O. B., 2023]

Let $\operatorname{char}\mathbb{F} \neq 2, 3$. An arbitrary automorphism φ of each of the finitary Mackey groups either lift to a ring automorphism of $A_\infty(V|W)$, or to

$$\varphi(g) = \psi((g^{-1})^t), \qquad g \in \mathrm{SL}_\infty(V|W),$$

where ψ is a ring anti-automorphisms of $A_\infty(V|W)$.

V. Mackey algebras and groups

This part is based on papers:

[1] O.B., Derivations of Mackey algebras,
Carpathian Mathematical Publications, **15**(2) (2023), 559-562. **Q2**

[2] O.B., Automorphisms of Mackey groups,
Bulletin of Taras Shevchenko National University of Kyiv,
Series: Physics & Mathematics, **2** (2023), 16-19. **Q4**

VI. Derivations of polynomial algebras in infinitely many variables

Now, we will discuss connections of Theorems 10 and 11 to derivations of polynomial algebras.

Assume $\mathrm{char}\,\mathbb{F} = 0$.

Let $A = \mathbb{F}[x_1, \dots, x_n]$ be the ***polynomial algebra*** in n variables.

Any derivation $d : A \to A$ may be written uniquely as

$$d = \sum_{i=1}^{n} f_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i}, \quad f_i \in A.$$

The set of all derivations of A is a Lie algebra under the commutator $[d_1, d_2] = d_1 d_2 - d_2 d_1$, denoted $W(n)$.

VI. Derivations of polynomial algebras in infinitely many variables

[F. Takens, 1973, T. Morimoto, 1976]

Every derivation of the Lie algebra $W(n)$ is inner.

[A. Rudakov, 1969]

Every automorphism of the Lie algebra $W(n)$ is of the form

$$\varphi(d) = \psi^{-1} \, d \, \psi, \quad \psi \in \mathrm{Aut}(\mathbb{F}[x_1, \ldots, x_n]).$$

New proofs of these theorems were given by V. Bavula (2013).
The similar result was later proved by H. Kraft and A. Regeta (2014) in group-theoretic terms.

VI. Derivations of polynomial algebras in infinitely many variables

Consider the ascending chain of polynomial subalgebras:

$$\mathbb{F}[x_1] \subset \mathbb{F}[x_1, x_2] \subset \cdots \subset \mathbb{F}[x_1, \dots, x_n] \subset \cdots .$$

We have the ascending chain of Lie algebras:

$$W(1) \subset W(2) \subset \cdots \subset W(n) \subset \cdots .$$

Let

$$W(\infty) = \bigcup_{n \geq 1} W(n).$$

Consider the countable set of variables $X = \{x_1, x_2, \dots\}$ and the polynomial algebra $\mathbb{F}[X]$. Then

$$W(\infty) \subset \text{Der}\,(\mathbb{F}[X]), \quad \text{where}$$

$$\text{Der}\,(\mathbb{F}[X]) = \left\{ \sum_{i=1}^{\infty} f_i(x_1, x_2, \dots) \frac{\partial}{\partial x_i} \;\middle|\; f_i \in \mathbb{F}[X] \right\}$$ with possibly infinite sums.

VI. Derivations of polynomial algebras in infinitely many variables

[D.Ž. Doković – K. Zhao, 1998]

Every derivation d of the Lie algebra $\mathcal{L} = \mathrm{Der}\,(\mathbb{F}[X])$ is **locally inner**, i.e. for every finite-dimensional subspace $V \subset \mathcal{L}$ there exists an element $x \in \mathcal{L}$ such that

$$d(v) = [x, v] \quad \text{for all } v \in V.$$

A stronger result holds as well.

Theorem 14 [O. B. – I. Kashuba, 2025]

Every derivation of the Lie algebra $\mathrm{Der}\,(\mathbb{F}[X])$ is **inner**.

VI. Derivations of polynomial algebras in infinitely many variables

Question

What about derivations of $W(\infty)$? Are they all inner? **No.**

Consider the Lie algebra

$$W_{\text{fin}} = \Big\{ \sum_{i=1}^{\infty} f_i(x_1, x_2, \dots) \frac{\partial}{\partial x_i} \mid f_i \in \mathbb{F}[X],$$

$$\text{each } x_i \text{ occurs in only finitely many } f_j,\ j \geq 1 \Big\}.$$

Then $W(\infty) \subset W_{\text{fin}} \subset \operatorname{Der} \mathbb{F}[X].$

Remark

The description of derivations of $W(\infty)$ is closer "in some sense" **to the description of derivations of locally matrix algebras.**

VI. Derivations of polynomial algebras in infinitely many variables

Now consider the **"tails"** of the polynomial algebra:

$$\mathbb{F}[x_n, x_{n+1}, \dots] \subset \mathbb{F}[X].$$

Clearly,

$$\bigcap_{n\geq 1} \mathbb{F}[x_n, x_{n+1}, \dots] = (0).$$

Hence, these **"tails" define a topology** on the vector space $\mathbb{F}[X]$ (though $\mathbb{F}[X]$ is not a topological algebra).

VI. Derivations of polynomial algebras in infinitely many variables

Therefore,

$$W_{\text{fin}} = \{\text{derivations of } \mathbb{F}[X] \text{ that are continuous in this topology}\}.$$

The algebra $W(\infty)$ is an ideal in W_{fin}. Hence, any element $a \in W_{\text{fin}}$ defines a derivation

$$d_a : W(\infty) \longrightarrow W(\infty), \quad d_a(x) = [x, a].$$

Theorem 15 [O.B. – I. Kashuba, 2025]

Every derivation of the Lie algebra $W(\infty)$ is of the form

$$d_a(x) = [x, a], \quad a \in W_{\text{fin}}.$$

In particular,

$$\text{Der}\left(W(\infty)\right) \cong W_{\text{fin}}.$$

VI. Derivations of polynomial algebras in infinitely many variables

Question [D.Ž. Doković – K. Zhao, 1998]

Is the Lie algebra $\mathrm{Der}\,(\mathbb{F}[X])$ simple?

Conjecture [O. B.]

$$\mathrm{Aut}\,\big(W(\infty)\big) \; = $$

$$\Big\{ \mathrm{ad}_\psi : d \mapsto \psi^{-1} d \psi \;\Big|\; \psi \in \mathrm{Aut}\,\big(\mathbb{F}[X]\big),\ \psi \text{ and } \psi^{-1} \text{ are both continuous} \Big\}.$$

This part is based on the paper:

[1] O.B., I.Kashuba, Derivations of Lie Algebras of Vector Fields in Infinitely Many Variables, arXiv:2507.04541 (2025).

This talk is based on papers:

1. O.B., B.Oliynyk, Primary decompositions of unital locally matrix algebras, *Bulletin of Mathematical Sciences*, **10**(1) (2020). **Q1**
2. O.B., B.Oliynyk, Unital locally matrix algebras and Steinitz numbers, *Journal of Algebra and Its Applications*, **19**(9) (2020). **Q2**
3. O.B., Derivations and automorphisms of locally matrix algebras, *Journal of Algebra*, **576** (2021), 1-26. **Q1**
4. O. B., Automorphism and derivations of algebras of infinite matrix, *Linear Algebra and Applications*, **650**(2) (2022), 42-59. **Q1**
5. O.B., W.Golubowski, B.Oliynyk, Ideals of general linear Lie algebras of infinite-dimensional vector spaces, *Proceedings of the American Mathematical Society*, **151** (2023), 467-473. **Q1**
6. O.B., Automorphisms of Mackey groups, *Bulletin of TSNU of Kyiv*, Series: Physics & Mathematics, **2** (2023), 16-19. **Q4**

This talk is based on papers:

7. O.B., Derivations of Mackey algebras, *Carpathian Mathematical Publications*, **15**(2) (2023), 559-562. **Q2**
8. O.B, A.Petravchuk, E.Zelmanov, Automorphisms and derivations of commutative and PI algebras, *Transactions of the American Mathematical Society*, **377**(2) (2024), 1335-1356. **Q1**
9. O.B, I.Kashuba, E.Zelmanov, On Lie isomorphisms of rings, *Mediterranean Journal of Mathematics* (2025) 22:80. **Q2**
10. O.B., On Clifford Algebras of Infinite Dimensional Vector Spaces, Proceedings XIV Ukraine Algebra Conference, *American Mathematical Society: Contemporary Mathematics* (2025). **Q3**
11. O.B., I.Kashuba, Derivations of Lie Algebras of Vector Fields in Infinitely Many Variables, arXiv:2507.04541 (2025).

THANKS FOR YOUR ATTENTION!

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